2001. Assuming non-replacement from the wording of the question,

$$p = \tfrac{26}{52} \times \tfrac{25}{51} \times \tfrac{26}{50} \times \tfrac{25}{49} = 0.0650 \text{ (3sf)}.$$

2002. (a) $\frac{8}{5} = 1 + \frac{3}{5}$.

(b) $\dfrac{2x+3}{x+1} \equiv \dfrac{2(x+1)+1}{x+1} \equiv 2 + \dfrac{1}{x+1}$.

——— Alternative Method ———

Using polynomial long division,

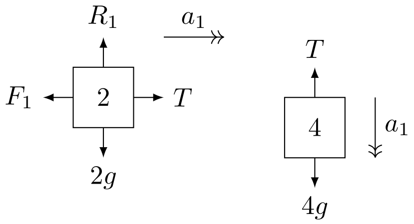
$$\begin{array}{r|l} & 2 \\ x+1 & 2x+3 \\ & -2x-2 \\ \hline & 1 \end{array}$$

The quotient is 2 and the remainder 1, so

$$\frac{2x+3}{x+1} \equiv 2 + \frac{1}{x+1}.$$

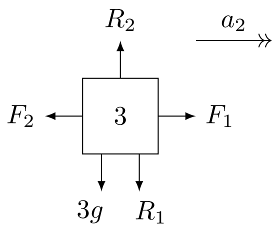
2003. (a) The relevant assumption is that the string is inextensible. This means that the 2 kg and 4 kg masses must have the same acceleration a_1. The 3 kg mass, which is not attached to the others by a string, may have a different acceleration a_2.

(b) Force diagrams for the connected blocks are:



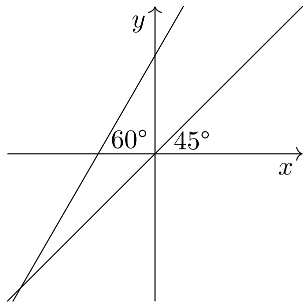
Vertically, $R_1 = 2g$, so that the maximum value of F_1 is $\mu R_1 = g$. Using this value, the equations of motion are $T - g = 2a_1$ and $4g - T = 4a_1$. Adding these gives $3g = 6a$, so $a_1 = \frac{1}{2}g$. Since this is positive, taking the maximum value for F_1 was correct.

The force diagram for the 3 kg block is



$R_2 = 5g$, so the maximum value of F_2 is $\frac{5}{2}g$. Using this value, $g - \frac{5}{2}g = 3a_2$. This gives a negative value for a_2, which tells us that F_2 will, in fact, be less than maximal, and that $a_2 = 0$.

2004. The lines make acute angles $\arctan\sqrt{3} = 60°$ and $\arctan 1 = 45°$ with the x axis.



Chasing these angles into the triangle formed by the lines and the x axis, the acute angle between the lines is $60° - 45° = 15°$.

2005. (a) For $\mathrm{F}(x)\,\mathrm{G}(x)$ to repeat, both of the functions F and G must do so. This first occurs at $\mathrm{lcm}(4,6) = 12$. Hence, the period is 12.

(b) Considering input transformations, the terms $\mathrm{F}(2x)$ and $\mathrm{G}(3x)$ both have period 2. Hence, their sum also has period 2.

2006. Any straight line through $(2,-2)$ has equation $y+2 = m(x-2)$, which gives $y = mx-2m-2$. For intersections with the curve, $x^2 - x = mx - 2m - 2$, which simplifies to

$$x^2 - (m+1)x + 2(m+1) = 0.$$

For a tangent, we require exactly one intersection, hence we set $\Delta = (m+1)^2 - 8(m+1) = 0$. This gives $m = -1$ or $m = 7$. Substituting these values back in and solving gives Q as $(0,0)$ or $(4,12)$.

——— Alternative Method ———

The gradient at $(a, a^2 - a)$ is $2a - 1$. The equation of the tangent is $y - a^2 + a = (2a-1)(x-a)$. We need this to go through $(2,-2)$, so

$$\begin{aligned} & -2 - a^2 + a = (2a-1)(2-a) \\ \implies & a^2 - 4a = 0 \\ \implies & a = 0, 4. \end{aligned}$$

Hence, the coordinates of Q are $(4,12)$.

2007. Writing the sum out longhand, we can solve as follows:

$$\begin{aligned} & \sum_{r=1}^{2} \frac{x^r}{1-x^r} = 0 \\ \implies & \frac{x}{1-x} + \frac{x^2}{1-x^2} = 0 \\ \implies & x(1+x) + x^2 = 0 \\ \implies & 2x^2 + x = 0 \\ \implies & x = 0, -\tfrac{1}{2}. \end{aligned}$$

For both of these, the fractions are well defined.

2008. Since the m-sided die has fewer faces, consider rolling it first. Whatever number shows can be matched: the probability that the n-sided die matches it is $\frac{1}{n}$.

— Alternative Method —

The possibility space is an $m \times n$ grid of outcomes. The successful ones are on the leading diagonal, which, since $m < n$, has length m. Therefore, the probability is $\frac{m}{mn} = \frac{1}{n}$.

2009. Using $\sin^2\theta + \cos^2\theta \equiv 1$,

$$\begin{aligned}\cos^2 36° &= 1 - \frac{10 - 2\sqrt{5}}{16}\\ &= \frac{6 + 2\sqrt{5}}{16}.\end{aligned}$$

Taking the positive square root and reciprocating,

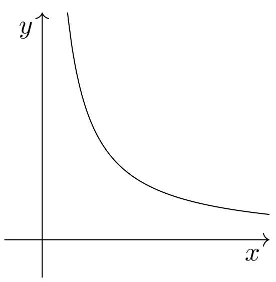
$$\sec 36° = \frac{4}{\sqrt{6 + 2\sqrt{5}}}.$$

We rationalise the denominator by multiplying top and bottom by its conjugate $\sqrt{6 - 2\sqrt{5}}$:

$$\begin{aligned}\sec 36° &= \frac{4\sqrt{6 - 2\sqrt{5}}}{4}\\ &= \sqrt{6 - 2\sqrt{5}}.\end{aligned}$$

2010. Differentiating twice, we get $f'(x) = 4x^3 + 2ax$ and then $f''(x) = 12x^2 + 2a$. If $a > 0$, as we are told, then the second derivative is positive everywhere. This is the same as saying that the first derivative is increasing everywhere. Hence, there can be at most one point at which $f'(x) = 0$. □

2011. Rearranging and squaring gives $y = \frac{1}{x}$, which is the standard reciprocal graph. However, negative values of x and y must be excluded, as the original equation is undefined. Hence, the graph is

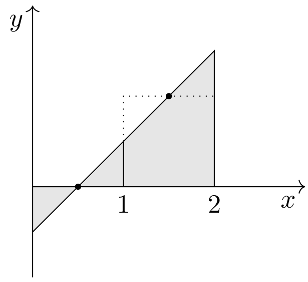


2012. (a) True: reflection in $y = x$.

(b) True: rotation by 180° around $(0, 1/2)$.

(c) True: the latter may be rewritten $y = (x-1)^2$, giving translation by vector $-2\mathbf{i}$.

2013. Since $y = f(x)$ is a straight line, the first integral tells us that the line passes through $(1/2, 0)$, and the second integral tells us that it passes through $(3/2, 1)$. (Consider the rectangles with the same area as the trapezia.)



Hence, $f(2) = 3/2$.

— Alternative Method —

Let $f(x) = ax + b$, where a, b are constants. The integral of f is $F(x) = \frac{1}{2}ax^2 + bx$. So,

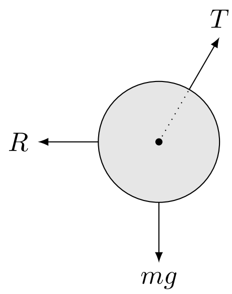
$$\begin{aligned}F(1) - F(0) &= 0,\\ F(2) - F(1) &= 1.\end{aligned}$$

These equations are

$$\begin{aligned}\tfrac{1}{2}a + b &= 0,\\ 2a + 2b &= 1.\end{aligned}$$

Solving, $a = 1$ and $b = -0.5$. So, $f(2) = 3/2$.

2014. The triangle formed is equilateral, so each string makes an angle of 60° with the horizontal. Due to the symmetry of the problem, the baubles cannot exert a vertical force on one another. So, the force diagram for the left-hand bauble is



Resolving vertically,

$$\begin{aligned}T\sin 60° - mg &= 0\\ \Longrightarrow T &= \tfrac{2\sqrt{3}}{3}mg.\end{aligned}$$

The same applies symmetrically for the second bauble: both tensions are $\frac{2\sqrt{3}}{3}mg$ N.

2015. For θ in degrees, each value θ must be scaled by $\frac{\pi}{180}$, i.e. converted into radians, before being put into $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. Hence, the approximation is

$$\cos\theta \approx 1 - \tfrac{1}{2}\left(\frac{\pi}{180}\theta\right)^2 \equiv 1 - \frac{\pi^2}{64800}\theta^2.$$

2016. Call the initial speed u. The vertical motion is

s	0
u	$u\sin 15°$
v	
a	$-g$
t	t

This gives $0 = (u\sin 15°)t - \frac{1}{2}gt^2$. Either $t = 0$, which is launch i.e. not the root we are after, or $0 = u\sin 15° - \frac{1}{2}gt$. Horizontally, $40 = (u\cos 15°)t$. Substituting the latter yields

$$\begin{aligned} & 0 = u\sin 15° - \tfrac{1}{2}g\frac{40}{u\cos 15°} \\ \Longrightarrow\ & u^2 \sin 15° \cos 15° = 20g \\ \Longrightarrow\ & u^2 = 784. \end{aligned}$$

Taking the positive square root gives $u = 28$ ms^{-1}.

2017. We need to show that $6\mathbf{i} + 8\mathbf{j} + q\mathbf{k}$ cannot have magnitude 9. By Pythagoras, that would require $6^2 + 8^2 + q^2 = 9^2$. But this simplifies to $q^2 = -19$, which has no real roots, as required.

— Alternative Method —

The vector is shortest with q set to zero. By 2D Pythagoras, the magnitude is then $\frac{10}{9} > 1$. For any other q, the vector will be longer, so there is no q such that the vector is of unit magnitude.

2018. Since the equation has exactly one real root, the discriminant must be zero:

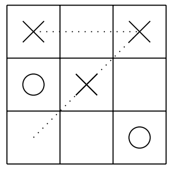
$$\begin{aligned} & (ad + bc)^2 - 4abcd = 0 \\ \Longrightarrow\ & a^2d^2 + 2abcd + b^2c^2 - 4abcd = 0 \\ \Longrightarrow\ & a^2d^2 - 2abcd + b^2c^2 \\ \Longrightarrow\ & (ad - bc)^2 = 0 \\ \Longrightarrow\ & ad = bc, \quad \text{as required.} \end{aligned}$$

2019. ① $\Longleftrightarrow$ ②. Both statements are equivalent to "The point (x, y) lies on the unit circle."

2020. For a function to be invertible, it must be one-to-one: multiple inputs cannot give the same output. But a quartic, as a polynomial of even degree, must have at least one turning point. Hence, over $\mathbb{R}$, it must have at least two inputs which produce the same output, which means it cannot be invertible over the domain $\mathbb{R}$. QED.

2021. (a) The normal distribution is symmetrical, so we have $-X \sim N(0, 1)$.
(b) This is the distribution from (a), translated by 1, so $1 - X \sim N(1, 1)$.
(c) 3 translates the mean, the stretch scales the variance by 2^2, giving $2X + 3 \sim N(3, 4)$.

2022. If × plays in the top right, then the game stands as follows, with ○ to move.



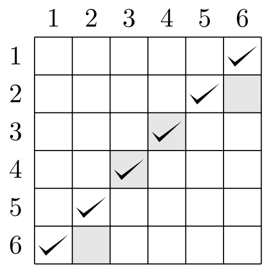
Two possible sets of three are created, as marked with the dotted lines. ○ cannot block both, and has therefore lost.

2023. The last equation tells us that either x or $(1 + y)$ is zero. But, since $xyz = 2$, x cannot equal zero. Therefore $y = -1$. The first two equations are now $x + z = 1$ and $xz = -2$. Solving simultaneously gives (x, y, z) as either $(-1, -1, 2)$ or $(2, -1, -1)$.

2024. We integrate the reciprocal function to $\ln|3x - 4|$, picking up $\frac{1}{3}$ by the reverse chain rule:

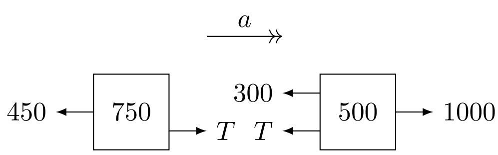
$$\int \frac{1}{3x - 4}\,dx = \tfrac{1}{3}\ln|3x - 4| + c.$$

2025. The possibility space is



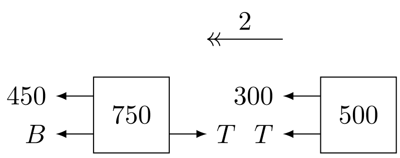
The fact "$XY = 12$" restricts the possibility space to the shaded squares above. Hence, it changes the probability of obtaining one of the ticked squares from $\frac{6}{36}$ to $\frac{2}{4}$. This is an increase.

2026. (a) The frictional force acting on the ground is one aspect of the driving force: its NIII pair, with magnitude 1000 N, acts forwards on the truck.



(b) NII for the system is $1000 - 300 - 450 = 1250a$, so $a = 0.2$ ms^{-2}. Then NII for the trailer is $T - 450 = 750 \times 0.2$, so $T = 600$ N.

(c) The new force diagrams are as follows:



The equation of motion for the truck is now $T + 300 = 500 \times 2$. This gives $T = 700$ N, which is an increase.

2027. Multiplying by x^2, the equation is $x^6 + x^3 + x^{-2}$, which is a quadratic in x^3. Its discriminant is $\Delta = 1^2 - 4 = -3 < 0$, which implies that it has no real roots.

— Alternative Method —

All three terms have even degree. Hence, they are all non-negative. Furthermore, x^{-2} is a reciprocal, so cannot be zero. Hence, the LHS is positive, so no real values of x satisfy the equation.

2028. For small angles, $\cos\theta$ approaches 1. Since $\frac{\sin\theta}{\theta}$ is sandwiched between $\cos\theta$ and 1, its value must also approach 1 for small angles. (This is called the *squeeze theorem.*) Hence, since the ratio of $\sin\theta$ and θ approaches 1, $\sin\theta \approx \theta$.

2029. Putting the fractions over a common denominator,

$$\begin{aligned}
&\frac{\sqrt{p}+\sqrt{q}}{\sqrt{p}-\sqrt{q}} + \frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}} \\
&\equiv \frac{\left(\sqrt{p}+\sqrt{q}\right)^2}{p-q} + \frac{\left(\sqrt{p}-\sqrt{q}\right)^2}{p-q} \\
&\equiv \frac{p+2\sqrt{pq}+q}{p-q} + \frac{p-2\sqrt{pq}+q}{p-q} \\
&\equiv \frac{2p+2q}{p-q}.
\end{aligned}$$

2030. (a) $\frac{d}{dx}\left(x^2\right) \equiv 2x$,

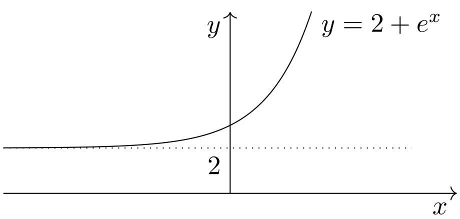
(b) By the chain rule, $\frac{d}{dx}\left(y^2\right) \equiv 2y\frac{dy}{dx}$,

(c) By the chain rule, $\frac{d}{dx}\left(e^{2y+1}\right) \equiv e^{2y+1} \cdot 2\frac{dy}{dx}$.

— Nota Bene —

Implicit differentiation, while it has a fancy name, is simply differentiation by the chain rule. The only difference is that the inside function of x, whose output is y, is not made explicit. Hence, the derivative of the inside function $y = \mathrm{f}(x)$ must be expressed using a generic $\frac{dy}{dx}$.

2031. The function need not have a local minimum at all. The function $\mathrm{f}(x) = 2 + e^x$ has domain $\mathbb{R}$ and range $(2, \infty)$, but is asymptotic to 2, rather than having a local minimum at 2:



2032. Every line through O is of the form $y = mx$, so we need only find the range of $\frac{1+a}{1-a}$, for $a \in \mathbb{R}$. Writing the top in terms of the bottom,

$$\frac{1+a}{1-a} \equiv \frac{2-(1-a)}{1-a} \equiv \frac{2}{1-a} - 1.$$

Since $\frac{2}{1-a}$ can take any value except zero, the lines can have any gradient but -1. Hence, the family contains all but one of the straight lines through the origin, as required.

2033. Name the couples A, B, C. Person A_1 can be placed arbitrarily. The probability that A_2 sits next to A_1 is $\frac{2}{5}$. This leaves four seats in a row. There are six orders of $B_1B_2C_1C_2$, of which two have both couples sitting together. So, the overall probability is $\frac{2}{5} \times \frac{2}{6} = \frac{2}{15}$.

— Alternative Method —

Person A_1 takes a seat. The probability that A_2 sits next to A_1 is $\frac{2}{5}$. Someone has to sit next to A_2. Call this person B_1. The probability that B_2 sits next to B_1 is $\frac{1}{3}$. If A_1, A_2, B_1, B_2 are all in a group, then C_1 and C_2 are automatically next to one another. So, the probability is $\frac{2}{5} \times \frac{1}{3} = \frac{2}{15}$.

2034. Multiplying out,

$$\begin{aligned}
&\mathrm{f}(x) = pqx^2 - px^4 \\
\Longrightarrow\ &\mathrm{f}'(x) = 2pqx - 4px^3.
\end{aligned}$$

A local maximum is a stationary point, so we know that $\mathrm{f}'(2) = 0$. This gives

$$\begin{aligned}
&4pq - 32p = 0 \\
\Longrightarrow\ &4p(q-8) = 0 \\
\Longrightarrow\ &p = 0 \text{ or } q = 8.
\end{aligned}$$

We know that $f(2) = 32 \neq 0$, so the latter case must hold. Using $\mathrm{f}(2) = 32$, we get $16p = 32$, so $p = 2$.

2035. This is a quadratic in x^{10}. The formula would work, or we can factorise to

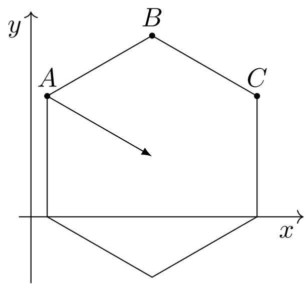
$$\begin{aligned}
&x^{20} - 1019x^{10} - 5120 = 0 \\
\Longrightarrow\ &(x^{10} - 1024)(x^{10} + 5) = 0 \\
\Longrightarrow\ &x^{10} = 1024, -5.
\end{aligned}$$

But x^{10} must be positive, ruling out -5. Hence, $x^{10} = 1024$. Taking the tenth root gives $x = \pm 2$.

2036. There are many ways in which this can be done, because scales, while they are used to measure mass in kg, actually measure contact reaction force. Anything that increases the contact force between feet and scales will raise the reading. This would occur in a lift accelerating upwards. It could also occur if the person was holding a light bungee cord attached to the floor, or was pushing again the ceiling.

2037. The four values have the same distribution. And the probability that any two are equal must be zero, since the interval contains infinitely many numbers. Hence, each is equally likely to be the largest: $p = \frac{1}{4}$.

2038. The scenario is



Any two vertices of a hexagon, together with the centre, form an equilateral triangle. Hence, we can translate the point A by the vector

$$\overrightarrow{BC} = \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}.$$

This gives the centre as $(2, 1)$.

2039. Writing b as $a^{\log_a b}$ and then using an index law to rearrange, we have a transformation

$$y = a^x \longmapsto y = a^{(\log_a b)x}.$$

This is an input transformation, replacing x by $(\log_a b)x$, which is a stretch in the x direction, length scale factor $1/\log_a b$. This can be rewritten as $\log_b a$. Since only the x dimension is scaled, this is also the area scale factor.

2040. (a) The median will be unaffected, because data have been removed symmetrically from the top and bottom of the sample. The specific values of those data do not affect the median.

(b) It is very likely, although not guaranteed, that the sample IQR will be reduced. In all natural cases, removing the highest and lowest data will reduce the spread. An exception would be a fabricated set of data of e.g. 40 zeros and 40 ones. The IQR is 1, and would remain so were the sample reduced to 30 zeros and 30 ones.

2041. The first derivative is

$$\frac{dy}{dx} = 1 + x^{-\frac{3}{2}}.$$

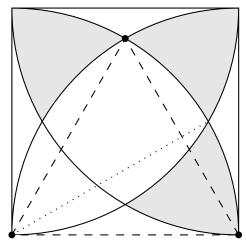
Since the domain $(0, \infty)$ is positive, this is positive everywhere; hence, the curve is increasing.

The second derivative is

$$\frac{d^2y}{dx^2} = -\tfrac{3}{2}x^{-\frac{5}{2}}.$$

Since the domain is $(0, \infty)$ positive, this is negative everywhere; hence, the curve is concave.

2042. The regions subtend the same angle if the dotted lines trisect the right angle. Hence, we need to show that each is 30°. This can be seen by noting that the three points marked below are, by dint of the circular arcs that connect them, equidistant from each other, and thereby form an equilateral triangle:



This gives three angles of 30° at the bottom-left vertex. So, the shaded areas all subtend the same angle, as required.

2043. Quoting the standard derivative,

$$f'(x) = \sec^2(x + k).$$

Substituting into the LHS of the DE,

$$\begin{aligned} & f'(x) - \big(f(x)\big)^2 \\ \equiv\ & \sec^2(x + k) - \tan^2(x + k). \end{aligned}$$

The Pythagorean identity $1 + \tan^2 x \equiv \sec^2 x$, if we replace x by $x + k$ and rearrange, is

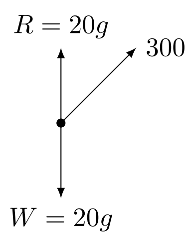
$$\sec^2(x + k) - \tan^2(x + k) \equiv 1.$$

Therefore, $f(x) = \tan(x + k)$ satisfies the DE.

— Nota Bene —

Here, we don't need the chain rule to find the derivative. Or rather, we do need it, but since the derivative of $x+k$ is 1, the chain rule has no effect. The general result is that, for an input translation $x \mapsto x+k$, the variable $x+k$ can be treated exactly like x in any calculus.

2044. (a) Since the girl exerts a force downwards and backwards on the ground, then by NIII the ground exerts a force upwards and forwards on her. Hence, the force diagram is



(b) Her weight and the reaction of magnitude $20g$ cancel. So, acceleration is in the direction of the 300 N force. In this direction, the equation of motion is $300 = 20a$, giving $a = 15$ ms^{-1}. Assuming that she begins at rest, $v = u + at$ gives $v = 15 \times 0.2 = 3$ ms^{-1} at 45° above the horizontal.

(c) As soon as she leaves the ground, she may be modelled as a projectile. Her initial vertical velocity is $3 \sin 45°$. So, for the whole flight, $s = ut + \frac{1}{2}at^2$ gives

$$0 = \tfrac{3\sqrt{2}}{2}t - \tfrac{1}{2}gt^2.$$

Solving this, we get $t = 0$ for take-off and $t = {}^{3\sqrt{2}}/_{g}$ for landing. In this time, she travels

$$\begin{aligned} s_x &= 3\cos 45° \times \tfrac{3\sqrt{2}}{g} \\ &= 0.91836... \end{aligned}$$

She jumps around 92 cm, just short of a metre.

2045. Expanding the brackets and splitting the sum up,

$$\begin{aligned} &\sum_{i=1}^{n} \left(x_i^2 - 2x_i\bar{x} + \bar{x}^2\right) \\ \equiv &\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} x_i\bar{x} + \sum_{i=1}^{n} \bar{x}^2. \end{aligned}$$

Since $\bar{x}$ is constant across the sum, this is

$$\sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + n\bar{x}^2.$$

Then, by the definition of the sample mean, we know that $\sum x_i = n\bar{x}$. This gives

$$\begin{aligned} &\sum_{i=1}^{n} x_i^2 - 2n\bar{x}^2 + n\bar{x}^2 \\ \equiv &\sum_{i=1}^{n} x_i^2 - n\bar{x}^2, \text{ as required.} \end{aligned}$$

2046. (a) Substituting,

$$\begin{aligned} & x^2(x+4) - x(x+4)^2 = 16 \\ \Longrightarrow\ & 4x^2 + 16x + 16 = 0 \\ \Longrightarrow\ & (x+2)^2 = 0. \end{aligned}$$

So, there is one intersection, at $(-2, 2)$.

(b) Since the factor of $(x+2)$ is squared, the root at $x = -2$ is a double root. This means that the line does not cross the curve, but touches it at a single point. Hence, $y = x + 4$ is tangent to the curve at $(-2, 2)$.

2047. Notate the set $A \setminus B$.

Set A is all x within 6 (inclusive) of the number 4; in interval set notation, this is $[-2, 10]$. Set B is all numbers less than or equal to 1, i.e. $(\infty, 1]$. So, we remove any numbers in B from A, giving $A \setminus B = (1, 10]$.

2048. Translating into algebra, $\text{f}''(y) - \text{g}''(y) = k$, for some constant k. Integrating with respect to y,

$$\text{f}'(y) - \text{g}'(y) = ky + b.$$

Integrating again, and renaming $\frac{1}{2}k$ as a,

$$\text{f}(y) - \text{g}(y) = ay^2 + by + c,$$

for some constants a, b, c.

The graphs $x = \text{f}(y)$ and $x = \text{g}(y)$ intersect where $\text{f}(y) = \text{g}(y)$, i.e. where $\text{f}(y) - \text{g}(y) = 0$. From the previous equation, this occurs when

$$ay^2 + by + c = 0.$$

This is a quadratic equation, which has at most two roots, as required.

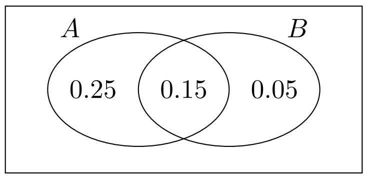
2049. (a) $k^{\frac{1}{2}x} \equiv (k^x)^{\frac{1}{2}} = \sqrt{y}$,

(b) $k^{\log_k 2 - \frac{1}{2}x} \equiv \dfrac{k^{\log_k 2}}{k^{\frac{1}{2}x}} \equiv \dfrac{2}{(k^x)^{\frac{1}{2}}} = \dfrac{2}{\sqrt{y}}$.

2050. Differentiating twice by the product rule,

$$\begin{aligned} &\frac{d^2}{dx^2}(e^x \sin x) \\ \equiv\ &\frac{d}{dx}(e^x \sin x + e^x \cos x) \\ \equiv\ &\frac{d}{dx}\big(e^x(\sin x + \cos x)\big) \\ \equiv\ &e^x(\cos x - \sin x) + e^x(\sin x + \cos x) \\ \equiv\ &2e^x \cos x. \end{aligned}$$

2051. The probabilities are as follows:



Both of the relevant probabilities are conditional on B, so we can restrict the possibility space to B. Since $0.15 > 0.05$, A is more likely to occur than not, given B. Algebraically, this is $\mathbb{P}(A \mid B) > \mathbb{P}(A' \mid B)$, as required.

2052. Differentiating by the chain rule,

$$\frac{d}{dx}\big(\ln(1 + e^x)\big) \equiv \frac{e^x}{1 + e^x}.$$

Since e^x is positive for all x, the denominator $1 + e^x$ is too. So, the first derivative is positive everywhere. Hence, $\ln(1 + e^x)$ is increasing for all $x \in \mathbb{R}$.

2053. The first counterexample is $3, 7, 11$, which is an AP with common difference 4.

2054. (a) This is true. Since f and g both have roots at b, we know that $\mathrm{f}(b) = \mathrm{g}(b) = 0$. This means that $\mathrm{f}(b) + \mathrm{g}(b) = 0$.

(b) This is not true. Consider any function f where $\mathrm{f}(0) \neq 0$. Then $\mathrm{fg}(b) = \mathrm{f}(\mathrm{g}(b)) = \mathrm{f}(0) \neq 0$ and $x = b$ is not a root.

(c) This is true. We know that $\mathrm{f}(a) = 0$, so it doesn't matter if $\mathrm{g}(a)$ is non-zero. Since $\mathrm{g}(a)$ is well-defined, $\mathrm{f}(a)\,\mathrm{g}(a) = 0$.

2055. Factorising the difference of two squares,

$$\begin{aligned} &\left(\sqrt{x+1} + \sqrt{x}\right)^2 - \left(\sqrt{x+1} - \sqrt{x}\right)^2 = 0 \\ \Longrightarrow\ &\left(2\sqrt{x}\right)\left(2\sqrt{x+1}\right) = 0. \end{aligned}$$

So either $\sqrt{x} = 0$ or $\sqrt{x+1} = 0$. The former gives $x = 0$. The latter gives $x = -1$, but this is ruled out by the domain of the square root function. So, the solution set is $\{0\}$.

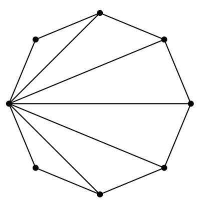
— Alternative Method —

Rearranging, the equation is

$$\begin{aligned} &\left(\sqrt{x+1} + \sqrt{x}\right)^2 = \left(\sqrt{x+1} - \sqrt{x}\right)^2 \\ \Longrightarrow\ &\sqrt{x+1} + \sqrt{x} = \pm\left(\sqrt{x+1} - \sqrt{x}\right). \end{aligned}$$

This gives $\sqrt{x} = 0$ or $\sqrt{x+1} = 0$. So, $x = 0$ or $x = -1$. But $\sqrt{x}$ is undefined for $x = -1$. So, the solution is $x = 0$.

2056. From any one vertex, $(n - 3)$ diagonals emerge, as the vertex itself and its two neighbours do not produce diagonals.



There are n vertices, giving $n(n - 3)$. However, in the calculation we have overcounted by a factor of 2, since each diagonal has two ends. Hence, the number of diagonals is $\frac{1}{2}n(n - 3)$. □

2057. At a point of inflection, the second derivative is zero and changes sign. But we don't need to worry about the sign change, since we are only putting a bound on the number of points of inflection.

If the polynomial graph has degree $n \geq 2$, then its second derivative has degree $n - 2 \geq 0$. Setting the second derivative to zero, we get a polynomial equation of degree $n - 2$. This can have at most $n - 2$ roots. QED.

— Nota Bene —

A polynomial equation of order 0 can, in fact, have infinitely many roots, if it is the trivial case $0 = 0$. But this is not possible in the above, as a quadratic graph $y = \mathrm{f}(x)$ must have a non-zero coefficient of x^2. In this case, the second derivative is a non-zero constant, which correctly gives no points of inflection.

2058. Having simplified the integrand, we use the reverse chain rule:

$$\begin{aligned} &\int_{-1}^{0} 1 - \frac{2x+1}{2x-1}\,dx \\ = &\int_{-1}^{0} \frac{2}{1-2x}\,dx \\ = &\Big[-\ln|1-2x|\Big]_{-1}^{0} \\ = &\ (-\ln 1) - (-\ln 3) \\ = &\ \ln 3, \text{ as required.} \end{aligned}$$

2059. We can choose one vertex arbitrarily, without loss of generality. Now, there remain five vertices $V_1, ..., V_5$, from which we must choose two. There are ${}^5\mathrm{C}_2 = 10$ ways of doing this. Only one of these, namely V_2 and V_4, yields an equilateral triangle. Hence, the probability is $1/10$.

— Nota Bene —

The standard phrase "without loss of generality" or "wlog" means "I am arbitrarily choosing one of various options, because I know, and am telling the reader, that it doesn't matter which I pick."

2060. The iteration $x_{n+1} = \mathrm{f}(x_n)$ has a fixed point where $x = \mathrm{f}(x)$ is satisfied. Hence, we require

$$b = 3b^2 + 2b + 4.$$

Rearranging to $3b^2 + b + 4 = 0$, the discriminant $\Delta = 1 - 4 \cdot 3 \cdot 4 = -47 < 0$. The equation has no (real) roots, so the iteration has no fixed points.

2061. (a) Setting $t = 0$, the rate is 412 litres per day.

(b) Setting $t = 280$, the rate at the end of the gestation period is modelled as

$$\frac{dV}{dt} = 365.101.$$

The percentage difference is

$$\frac{365 - 412}{412} = -11.4\% \text{ (1dp)}.$$

(c) To find the total production, we integrate the rate of production:

$$\begin{aligned} V &= \int_0^{280} 421 - 7.13 \times 10^{-4} t^2 \, dt \\ &= \left[421t - \tfrac{1}{3} 7.13 \times 10^{-4} t^3\right]_0^{280} \\ &= 112663. \end{aligned}$$

Take the rate of methane production in a non-pregnant cow to be a constant 412 litres/day. This gives a baseline of $421 \times 280 = 117880$. So, the percentage difference is

$$\frac{112663 - 117880}{117880} = -4.4\% \text{ (1dp)}.$$

2062. To generate $18x^2$, we need a squared term of $2(3x+2)^2$. This gives $24x$ as a by-product, where we need $72x$. So, the next term is $16(3x+2)$. We currently have $8 + 32 = 40$ as the constant term, requiring -44. So,

$$18x^2 + 72x - 4 \equiv 2(3x+2)^2 + 16(3x+2) - 44.$$

——— Alternative Method ———

Let $u = 3x + 2$, so $x = \frac{1}{3}(u-2)$. Substituting in,

$$\begin{aligned} &18x^2 + 72x - 4 \\ =\ &18\left(\tfrac{1}{3}(u-2)\right)^2 + 72\left(\tfrac{1}{3}(u-2)\right) - 4 \\ \equiv\ &2u^2 + 16u - 44 \\ =\ &2(3x+2)^2 + 16(3x+2) - 44. \end{aligned}$$

2063. We know that $xy = k$ for some constant k, so $y = kx^{-1}$. This gives $y' = -kx^{-2}$. Evaluating this at $x = 3$ and $x = 1$, the ratio in question is

$$-\frac{k}{9} : -\frac{k}{1}.$$

Multiplying both sides by $-\frac{9}{k}$, this is $1 : 9$.

2064. To find the mean, we sum over $i = 1, ..., n$ and divide by n. This is

$$\frac{1}{n} \sum_{i=1}^{n} \left(x_i^2 - \bar{x}^2\right).$$

Such a sum distributes over the subtraction, which means that it can be expressed as

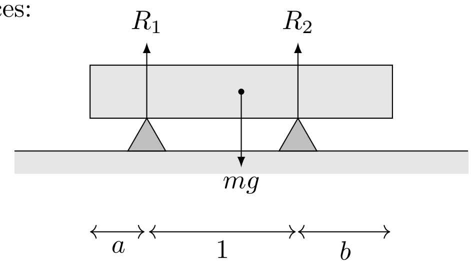
$$\frac{1}{n} \left(\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} \bar{x}^2 \right).$$

The mean $\bar{x}$ is constant across the right-hand sum. So, we have n copies of it added together, giving

$$\frac{1}{n} \left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 \right).$$

This is precisely the formula for the variance, in the form $\frac{1}{n} S_{xx}$. So, the mean value is s^2.

2065. Forces:



Taking the ratio values as lengths, the block is $1 + a + b$ long. So, the COM is $\frac{1}{2}(1 + a + b)$ from each end. Taking moments around the LH end,

$$aR_1 + (a+1)R_2 - mg\tfrac{1}{2}(1 + a + b) = 0.$$

We also have vertical equilibrium: $R_1 = mg - R_2$. Substituting this in,

$$a(mg - R_2) + (a+1)R_2 - mg\tfrac{1}{2}(1 + a + b) = 0$$

This can be simplified to

$$\begin{aligned} &amg + R_2 - \tfrac{1}{2}mg - \tfrac{1}{2}amg - \tfrac{1}{2}bmg = 0 \\ \Longrightarrow\ &R_2 = \tfrac{1}{2}mg\big(1 - (a-b)\big). \end{aligned}$$

Substituting back into $R_1 = mg - R_2$ gives

$$R_1 = \tfrac{1}{2}mg\big(1 + (a-b)\big), \text{ as required.}$$

2066. (a) For x-axis intercepts, we can factorise directly:

$$\begin{aligned} &\left(\tfrac{1}{3}x + 2\right)^3 - \left(\tfrac{1}{3}x + 2\right) = 0 \\ \Longrightarrow\ &\left(\tfrac{1}{3}x + 2\right)\left(\left(\tfrac{1}{3}x + 2\right)^2 - 1\right) = 0 \\ \Longrightarrow\ &\tfrac{1}{3}x + 2 = 0, \pm 1 \\ \Longrightarrow\ &x = -9, -6, -3. \end{aligned}$$

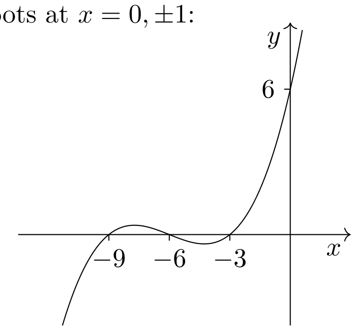
(b) Differentiating by the chain rule,

$$\begin{aligned} &y = \left(\tfrac{1}{3}x + 2\right)^3 - \left(\tfrac{1}{3}x + 2\right) \\ \Longrightarrow\ &\frac{dy}{dx} = 3 \cdot \tfrac{1}{3}\left(\tfrac{1}{3}x + 2\right)^2 - \tfrac{1}{3}. \end{aligned}$$

For SPs,

$$\begin{aligned} &\left(\tfrac{1}{3}x + 2\right)^2 - \tfrac{1}{3} = 0 \\ \Longrightarrow\ &\tfrac{1}{3}x + 2 = \pm\tfrac{\sqrt{3}}{3} \\ \Longrightarrow\ &x = -6 \pm \sqrt{3}. \end{aligned}$$

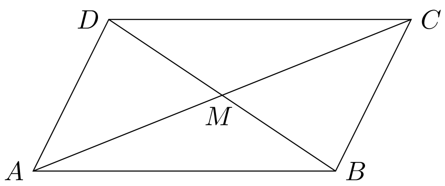
(c) The graph is an input-transformed version of $y = x^3 - x$, which is the usual positive cubic with roots at $x = 0, \pm 1$:



2067. Since the diagonals are perpendicular, we can place them on the x and y axes. Put A at $(a, 0)$ and B at $(0, b)$. Since $\overrightarrow{AB} = \overrightarrow{DC}$, C and D must be at $(-a, 0)$ and $(0, -b)$. Every side now has length $\sqrt{a^2 + b^2}$, so the shape is a rhombus. □

——— Alternative Method ———

Assume, for a contradiction, that the diagonals of a parallelogram are perpendicular, and the shape is not a rhombus:



The diagonals cross at M, the midpoint of AC and of BD. Since $|AB| \neq |AD|$, the cosine rule tells us that $\angle DMA \neq \angle BMA$. So, the diagonals are not perpendicular. This is a contradiction. Hence, if the diagonals of a parallelogram are perpendicular, then the shape must be a rhombus. □

2068. Assume, for a contradiction, that the odd number $4p^2 + 1$ is the square of an odd number of the form $2k+1$, where $k \in \mathbb{N}$. Hence $4k^2 + 4k + 1 = 4p^2 + 1$. This can be simplified to $k(k+1) = p^2$. Since the only prime factor of p^2 is p, this gives two options:

- $k = 1$, in which case $k + 1 = 2$ and $p = \sqrt{2}$,
- $k = p$, in which case $k + 1 = p$.

In both cases, there is a contradiction. Hence, $4p^2 + 1$ cannot be a perfect square. □

——— Alternative Method ———

Consider $4p^2 + 1 = n^2$, for $n \in \mathbb{N}$. Rearranging, we have a difference of two squares:

$$4p^2 = (n+1)(n-1).$$

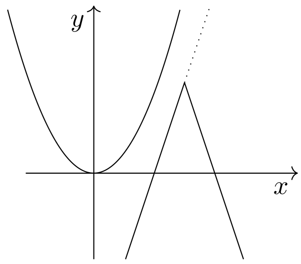
The LHS has a factor of 4. Hence, $(n+1)$ and $(n-1)$ must both be even. Furthermore, since they differ by 2, one must be a multiple of 4. So, the RHS has a factor of 8. Hence, p is even. It is prime, so $p = 2$. But this gives $4p^2+1 = 17$, which is not a perfect square. So, there are no natural numbers n for which $4p^2 + 1 = n^2$. □

2069. If the longest run of heads of tails is to be 2, then there are eight successful outcomes:

HTHT	HHTH	HTHH	HTTH
THTH	TTHT	THTT	THHT

There are 16 outcomes in the possibility space, all equally likely, so the probability is $\frac{1}{2}$.

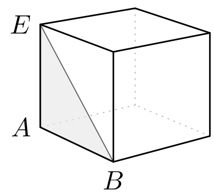
2070. The boundary equations are $y = 2 - 3|x - 2|$ and $y = x^2$. These are sketched below.



There are no points simultaneously above the parabola and below the mod graph. This may be proved by attempting to solve $y = 3x - 4$ (shown dotted above) and $y = x^2$ simultaneously. This gives $x^2 - 3x + 4 = 0$, which has $\Delta = -7$.

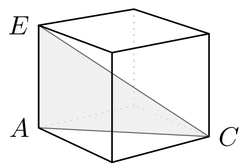
2071. In each case the relevant triangle is right-angled, so we can use basic trigonometry:

(a) AB and AE are edges of length 1, while BE is a face diagonal of length $\sqrt{2}$.



Hence, $\angle ABE = \arcsin \frac{1}{\sqrt{2}} = 45°$.

(b) AE is an edge of length 1, and CE is a space diagonal of length $\sqrt{3}$.



So, $\angle ACE = \arcsin \frac{1}{\sqrt{3}}$.

2072. Since $A = \pi r^2$, we know that $\frac{dA}{dr} = 2\pi r$. Then, the chain rule gives

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 2\pi r \times 5.$$

Evaluating this at $r = 10$ gives the rate of change of area as 100π cm^2/s.

2073. Writing the sum out longhand, and then quoting the exact values of the sine function,

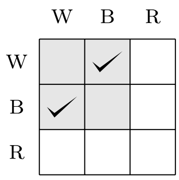
$$\begin{aligned}&\sum_{k=1}^{3} \operatorname{cosec}^2 \frac{\pi k}{4} \\ &= \frac{1}{\sin^2 \frac{\pi}{4}} + \frac{1}{\sin^2 \frac{\pi}{2}} + \frac{1}{\sin^2 \frac{3\pi}{4}} \\ &= \left(\frac{2}{\sqrt{2}}\right)^2 + 1 + \left(\frac{2}{\sqrt{2}}\right)^2 \\ &= 5.\end{aligned}$$

2074. The possibility space is a 3×3 grid of nine equally likely outcomes.

(a) There are four successful outcomes, so $p = \frac{4}{9}$.

(b) There are two successful outcomes, so $p = \frac{2}{9}$.

(c) Restricting to the shaded area according to the condition given, with the successful outcomes ticked, the possibility space is:

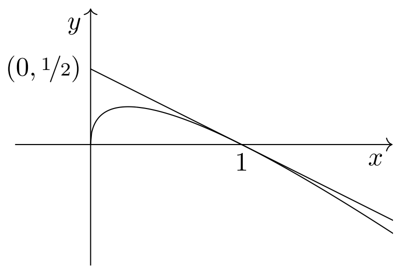


The probability is $\frac{2}{4} = \frac{1}{2}$.

2075. Solving for roots, we get $\sqrt{x} = x$. Squaring yields $x = x^2$, so $x = 0, 1$. Differentiating,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - 1.$$

Evaluating at $(1, 0)$, $m = -\frac{1}{2}$, so the tangent line is $y = -\frac{1}{2}x + \frac{1}{2}$. We cannot evaluate the gradient at $x = 0$, however, as division by zero is undefined. This means that the tangent is in the y direction at the origin; hence, it is the y axis.



Therefore the tangents meet at the y intercept of the other tangent, which is $(0, 1/2)$, as required.

2076. The squared difference of two squares factorises as $(x-1)^2(x+1)^2$. This is then a common factor, which we can take out. This yields

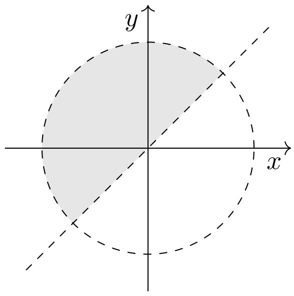
$$(x-1)^2(x+1)^2\big((x-1)+1\big) = 0.$$

Hence, $x = 0, \pm 1$.

2077. Since four is half of eight, every other face must be coloured, as on a chessboard. This means that the arrangement is dictated by the state of one of its faces. Hence, there are two successful outcomes. The total number of outcomes is the number of choices of 4 faces from 8, which is ${}^8\mathrm{C}_4 = 70$. Hence, the probability is $\frac{1}{35}$.

2078. The first derivative $\frac{dy}{dx}$ is a gradient formula, i.e. it produces values of the gradient as outputs. But we don't want the general formula, we want a specific gradient at the origin. Hence, the student should *evaluate* $\frac{dy}{dx}$ at $x = 0$, rather than using it as it is. This gives $m = 1$, so the correct equation of the tangent is $y = x$.

2079. Since $y = x$ is a diameter of the circle $x^2 + y^2 = 4$, the region required is a semicircle of radius 2. In each case, the boundary equation is not included:



2080. Differentiating with respect to u gives $\frac{dt}{du} = \sec^2 u$. Reciprocating both sides and using the definition of sec, this is $\frac{du}{dt} = \cos^2 u$. Lastly, we can use the first Pythagorean trig identity to yield the required result: $\frac{du}{dt} = 1 - \sin^2 u$.

2081. Taking the square to have unit side length, with the origin at the bottom left, let the point inside the square have coordinates (x, y). Then the lower and upper unshaded triangles have areas $\frac{1}{2}y$ and $\frac{1}{2}(1-y)$ respectively. These sum to $\frac{1}{2}$, which leaves half of the square shaded. $\square$

2082. Differentiating implicitly, we use the chain rule:

$$\begin{aligned} &\tfrac{d}{dt}\big(x^2 + y^2\big) + \tfrac{d}{dt}\big(x^2 - y^2\big) \\ &\equiv 2x\tfrac{dx}{dt} + 2y\tfrac{dx}{dt} + 2x\tfrac{dx}{dt} - 2y\tfrac{dx}{dt} \\ &\equiv 4x\tfrac{dx}{dt}. \end{aligned}$$

2083. (a) The variance is given by

$$s_x^2 = \frac{\sum x_i^2 - n\bar{x}^2}{n}.$$

We can rearrange this to

$$\sum x_i^2 = n\bar{x}^2 + ns_x^2.$$

Also, the mean $\bar{y}$ is given by $\frac{1}{n}\Sigma y_i$, which we are told is $\frac{1}{n}\Sigma x_i^2$. Combining our results,

$$\begin{aligned} \bar{y} &= \tfrac{1}{n}\sum x_i^2 \\ &= \tfrac{1}{n}(n\bar{x}^2 + ns_x^2) \\ &= \bar{x}^2 + s_x^2. \end{aligned}$$

(b) To calculate the variance of y, we would need to know Σy_i^2, which is Σx_i^4. But this info is not contained in either the mean or standard deviation; it not given in the question.

2084. As a limit, the derivative $\big(a\,\mathrm{f}(x) + b\,\mathrm{g}(x)\big)'$ is

$$\begin{aligned}&\lim_{h\to 0} \frac{a\,\mathrm{f}(x+h) + b\,\mathrm{g}(x+h) - (a\,\mathrm{f}(x) + b\,\mathrm{g}(x))}{h}\\ \equiv\ &\lim_{h\to 0} a\frac{\mathrm{f}(x+h) - f(x)}{h} + b\frac{\mathrm{g}(x+h) - \mathrm{g}(x)}{h}\\ \equiv\ & a\lim_{h\to 0} \frac{\mathrm{f}(x+h) - f(x)}{h} + b\lim_{h\to 0}\frac{\mathrm{g}(x+h) - \mathrm{g}(x)}{h}\\ \equiv\ & a\,\mathrm{f}'(x) + b\,\mathrm{g}'(x).\end{aligned}$$

Therefore, the differential operator is linear.

2085. Writing the sum longhand,

$$\begin{aligned}&\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3} = 0\\ \Longrightarrow\ & (x-2)(x-3) + (x-1)(x-3)\\ &\qquad + (x-1)(x-2) = 0\\ \Longrightarrow\ & 3x^2 - 12x + 11 = 0\\ \Longrightarrow\ & x = 2 \pm \tfrac{1}{\sqrt{3}}.\end{aligned}$$

2086. Solving for intersections,

$$\begin{aligned}& x^3 + x^2 = x^3 + x + 6\\ \Longrightarrow\ & x^2 - x - 6 = 0\\ \Longrightarrow\ & x = -2, 3.\end{aligned}$$

Over the domain $[-2, 3]$, the curve $y = x^3 + x + 6$ is above $y = x^3 + x^2$, as shown by the y intercepts. Hence, the area enclosed is

$$\begin{aligned}&\int_{-2}^{3} (x^3 + x + 6) - (x^3 + x^2)\,dx\\ =\ & \int_{-2}^{3} -x^2 + x + 6\,dx\\ =\ & \Big[-\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + 6x\Big]_{-2}^{3}\\ =\ & \tfrac{27}{2} - \big(-\tfrac{22}{3}\big)\\ =\ & \tfrac{125}{6}, \text{ as required.}\end{aligned}$$

2087. Assume, for a contradiction, that there does exist a smallest positive rational p/q, for $p, q \in \mathbb{N}$. Then $p/2q$ is a smaller positive rational number, which contradicts the initial assumption. Hence, there is no smallest positive rational number. □

2088. One of the brackets must be zero. The first can't be, however, since 2^x is always positive. So, we need $3^{2x} + 3^x - 5 = 0$. This is a quadratic in 3^x, which doesn't factorise. So, we use the formula, giving

$$3^x = \frac{-1 \pm \sqrt{21}}{2}.$$

One of these values is negative, so won't give a root. Taking logs with the positive value,

$$\begin{aligned}x &= \log_3 \frac{-1 + \sqrt{21}}{2}\\ &= 0.530610...\\ &= 0.531 \text{ (3sf)}.\end{aligned}$$

2089. (a) Setting the first derivative to zero,

$$3x^2 - 18x + 26 = 0,$$

This has solution $x = 2.42, 3.58$ (3sf).

(b) The central root lies between the two SPs. So, since it is an integer, it must be $x = 3$. Setting $\mathrm{f}(3) = 0$ gives $k = -24$.

2090. The second statement is false. If $\sin\beta = 0$, then β could be 0 or π radians. The former case gives $\cos\beta = 1$, but the latter gives $\cos\beta = -1$.

2091. The nth term of this sequence is

$$\begin{aligned}u_n &= 4 + 3(n-1)\\ &\equiv 1 + 3n.\end{aligned}$$

The 40th term is 121. Setting $1 + 3n = 100$, we get $n = 33$. Hence, terms

$$u_{33}, u_{34}, ..., u_{40}$$

have three digits. This is 8 out of 40 terms, so the probability is $1/5$.

2092. Writing as a quadratic in a, we have

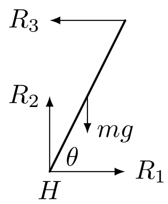
$$9b^{-\frac{1}{2}}a^2 - 12a + 4b^{\frac{1}{2}} = 0.$$

Using the quadratic formula,

$$\begin{aligned}a &= \frac{12 \pm \sqrt{4\cdot 9\cdot 4}}{18b^{-\frac{1}{2}}}\\ &\equiv \frac{(12 \pm 12)b^{\frac{1}{2}}}{18}.\end{aligned}$$

Since $a > 0$, we take the positive root: $a = \dfrac{4\sqrt{b}}{3}$.

2093. (a) Expressing the contact force at the hinge in components, the forces on the rod are:



(b) Calling the length of the rod l,

$$\begin{aligned}\updownarrow\ &: R_2 = mg\\ \leftrightarrow\ &: R_1 = R_3\\ \overset{\frown}{H}\ &: mgl\cos\theta - 2mgR_3 l\sin\theta = 0.\end{aligned}$$

From $\overset{\frown}{H}$, we get

$$R_3 = \tfrac{1}{2}mg\cot\theta = R_1.$$

So, the magnitude of the hinge force is

$$\begin{aligned} F &= \sqrt{R_1^2 + R_2^2} \\ &= mg\sqrt{\tfrac{1}{4}\cot^2\theta + 1} \\ &= \tfrac{1}{2}mg\sqrt{\cot^2\theta + 4}. \end{aligned}$$

We then use the third Pythagorean identity $\cot^2\theta \equiv \operatorname{cosec}^2\theta - 1$, which gives

$$F = \tfrac{1}{2}mg\sqrt{\operatorname{cosec}^2\theta + 3}, \text{ as required.}$$

2094. The statement is false. It is not significant that the probability of a *specific* outcome $X = k$ is below 5%. It is the *cumulative* probability of an outcome as extreme as k that is relevant. Depending on the hypotheses, the statement should be of the form

$$\mathbb{P}(X \geq k) < \tfrac{1}{20}.$$

— Nota Bene —

The above can be visualised with reference to e.g.

$$X \sim B(10000, 1/2).$$

In this case, the expected value is $\mathbb{E}(X) = 5000$. However, the probability that $X = 5000$ is small:

$$\mathbb{P}(X = 5000) \approx 0.008.$$

The exact result 5000 is unlikely, yes, but that wouldn't lead a statistician to doubt the fairness of a coin that produced 5000 heads in 10000 tosses.

2095. (a) This is true. Whatever the function f, F(0) is an integral from $t = 0$ to $t = 0$, which must have value 0.

(b) This is not true. A counterexample is $\mathrm{f}(t) = t$, which gives $\mathrm{F}(x) = \frac{1}{2}x^2$. So,

$$\mathrm{F}(ax + b) = \tfrac{1}{2}(ax + b)^2 \not\equiv a\left(\tfrac{1}{2}x^2\right) + b.$$

(c) This is true. Since F is the anti-derivative, i.e. indefinite integral of f, differentiating it yields f again.

2096. Differentiating by the chain rule,

$$\frac{dy}{dx} = e^{\sin 2x} \cdot \cos 2x \cdot 2.$$

Evaluating at $x = 0$, this is $e^0 \cdot 1 \cdot 2$, which is 2. The coordinate at this point is $(0, 1)$, so the tangent has equation $y = 2x + 1$.

2097. Clearly, both statements are true when $x = y$. So, we need only consider $x = -y$. In this case $|x| = |y|$, as the mod sign removes the negative. But $x^3 \neq y^3$, since cubing preserves the negative. So, the implication is $x^3 = y^3 \implies |x| = |y|$, but not vice versa.

2098. The intersections are at $x^2 = kx - x^2$, which is $x(2x - k) = 0$. So $x = 0$ or $x = \frac{1}{2}k$. The area enclosed is given, then, by the definite integral

$$\int_0^{\frac{1}{2}k} kx - 2x^2 \, dx.$$

Equating to $\frac{8}{3}$,

$$\begin{aligned} &\left[\tfrac{1}{2}kx^2 - \tfrac{2}{3}x^3\right]_0^{\frac{1}{2}k} = \tfrac{8}{3} \\ \Longrightarrow\ &\tfrac{1}{8}k^3 - \tfrac{1}{12}k^3 = \tfrac{8}{3} \\ \Longrightarrow\ &k^3 = 64 \\ \Longrightarrow\ &k = 4. \end{aligned}$$

2099. (a) $219 = 3 \times 73$.

(b) Using the binomial expansion, the LHS is

$$2a^3\sqrt{2} + 6a^2b\sqrt{3} + 9ab^2\sqrt{2} + 3b^3\sqrt{3}.$$

Equating coefficients of $\sqrt{2}$ and $\sqrt{3}$,

$$\begin{aligned} 2a^3 + 9ab^2 &= 486 \\ 6a^2b + 3b^3 &= 219. \end{aligned}$$

Dividing the latter by 3 and factorising,

$$b(2a^2 + b^2) = 73.$$

73 is prime, so, since b is an integer less than $(2a^2 + b^2)$, it must be 1. This gives $a = 6$.

2100. (a) $\mathrm{f}'(x) = 9x^2 - 22x - 11$.

(b) The N-R iteration is

$$x_{n+1} = x_n - \frac{3x_n^3 - 11x_n^2 - 11x_n - 14}{9x_n^2 - 22x_n - 11}.$$

Running the iteration with almost any starting point yields $x = 4.6666$, so we propose $x = \frac{14}{3}$ as a root.

(c) By the factor theorem $(3x - 14)$ should be a factor. It is, as follows:

$$\begin{aligned} &3x^3 - 11x^2 - 11x - 14 \\ \equiv\ &(3x - 14)(x^2 + x + 1). \end{aligned}$$

The quadratic has discriminant $\Delta = -3$, so has no real roots. Hence, the full solution set is $\{14/3\}$.

— End of 21st Hundred —